Practical Methods for Optimal Control Using Nonlinear Programming

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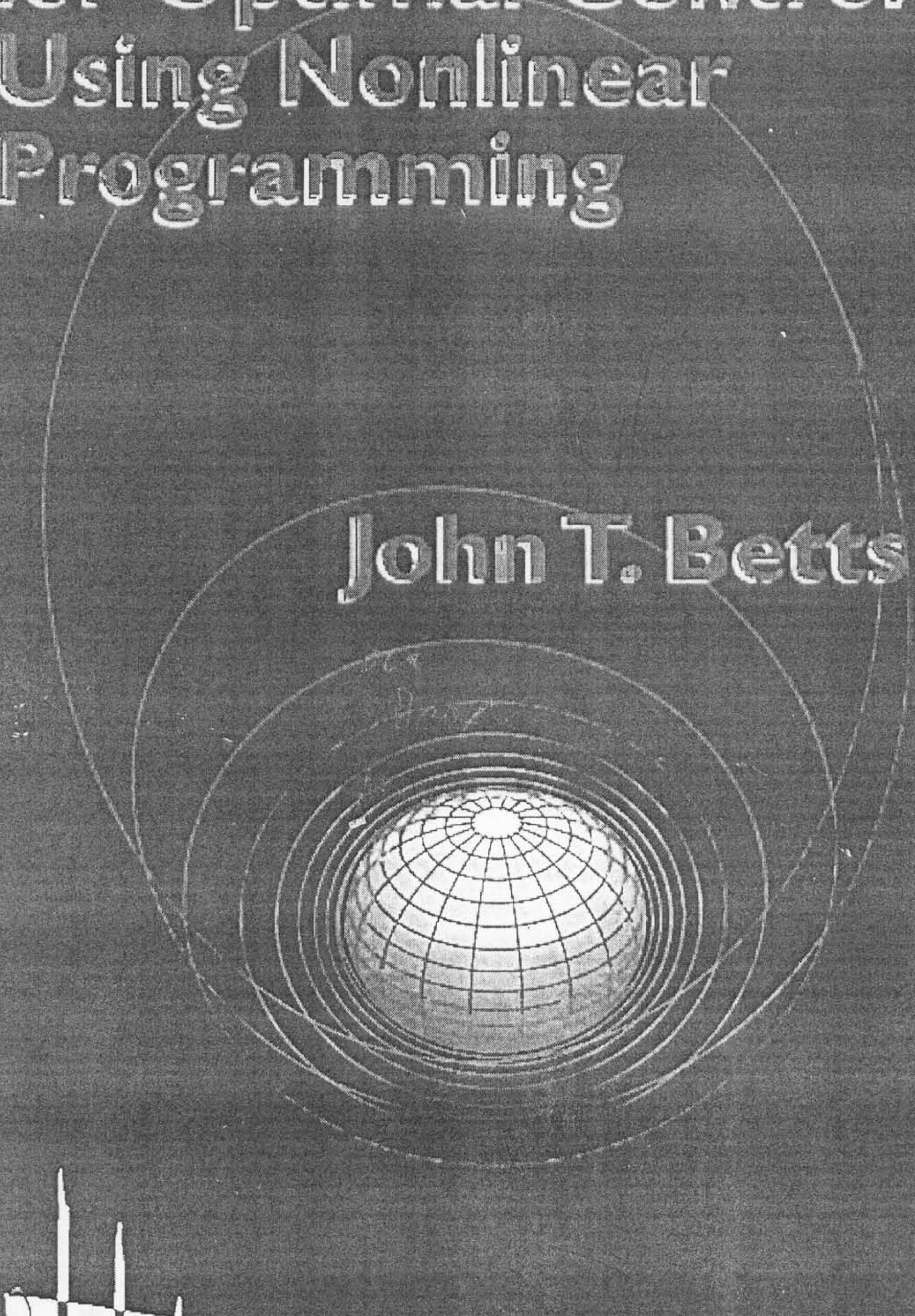

Advances in Design and Control

5.2 Minimum Time to Climb

(1)

Example 5.3. The original minimum time to climb problem was presented by Bryson et al. [34] and has been the subject of many analyses since then. Although the problem is not nearly as difficult to solve as the shuttle reentry examples, it is included here because it illustrates the treatment of tabular data. The basic problem is to choose the optimal control function $\alpha(t)$ (the angle of attack) such that an airplane flies from a point on a runway to a specified final altitude as quickly as possible. In its simplest form, the planar motion of the aircraft is described by the following set of ODEs:

$$\dot{h} = v \sin\gamma, \tag{5.11}$$

$$\dot{v} = \frac{1}{m}[T(M,h)\cos(\alpha) - D] - \frac{\mu}{(R_e+h)^2}\sin\gamma, \tag{5.12}$$

$$\dot{\gamma} = \frac{1}{mv}[T(M,h)\sin(\alpha) + L] + \cos\gamma\left[\frac{v}{(R_e+h)} - \frac{\mu}{v(R_e+h)^2}\right], \tag{5.13}$$

$$\dot{w} = \frac{-T(M,h)}{I_{sp}}, \tag{5.14}$$

where h is the altitude (ft), v the velocity (ft/sec), γ the flight path angle (rad), w the weight (lb), $m = w/g_0$ the mass, μ the gravitational constant, and R_e the radius of the earth. Furthermore, the simple bounds

$$0 \le h \le 69000. \text{ (ft)}, \qquad 1 \le v \le 2000. \text{ (ft)},$$
$$-89 \text{ (deg)} \le \gamma \le 89 \text{ (deg)}, \qquad 0 \le w \le 45000 \text{ (lb)},$$
$$-20 \text{ (deg)} \le \alpha \le 20 \text{ (deg)}$$

are also imposed.

The aerodynamic forces on the vehicle are defined by the expressions

$$D = \frac{1}{2}C_D S\rho v^2, \tag{5.15}$$

$$L = \frac{1}{2}C_L S\rho v^2, \tag{5.16}$$

$$C_L = c_{L\alpha}(M)\alpha, \tag{5.17}$$

$$C_D = c_{D0}(M) + \eta(M)c_{L\alpha}(M)\alpha^2, \tag{5.18}$$

where D is the drag, L is the lift, C_L and C_D are the aerodynamic lift and drag coefficients, respectively, with S the aerodynamic reference area of the vehicle, and ρ is the atmospheric density. Although the results presented here use a cubic spline approximation to the 1962 Standard Atmosphere, qualitatively similar results can be achieved with a simple exponential approximation to $\rho(h)$ (cf. Example 5.1). The following constants complete the definition of the problem:

$$h(0) = 0. \text{ (ft)}, \qquad h(t_F) = 65600.0 \text{ (ft)},$$
$$v(0) = 424.260 \text{ (ft/sec)}, \qquad v(t_F) = 968.148 \text{ (ft/sec)},$$
$$\gamma(0) = 0. \text{ (rad)}, \qquad \gamma(t_F) = 0. \text{ (rad)},$$
$$w(0) = 42000.0 \text{ (lb)}, \qquad S = 530. \text{ (ft}^2\text{)},$$

M	Thrust $T(M,h)$ (thousands of lb) Altitude h (thousands of ft) 0	5	10	15	20	25	30	40	50	70
0.0	24.2									
0.2	28.0	24.6	21.1	18.1	15.2	12.8	10.7			
0.4	28.3	25.2	21.9	18.7	15.9	13.4	11.2	7.3	4.4	
0.6	30.8	27.2	23.8	20.5	17.3	14.7	12.3	8.1	4.9	
0.8	34.5	30.3	26.6	23.2	19.8	16.8	14.1	9.4	5.6	1.1
1.0	37.9	34.3	30.4	26.8	23.3	19.8	16.8	11.2	6.8	1.4
1.2	36.1	38.0	34.9	31.3	27.3	23.6	20.1	13.4	8.3	1.7
1.4		36.6	38.5	36.1	31.6	28.1	24.2	16.2	10.0	2.2
1.6				38.7	35.7	32.0	28.1	19.3	11.9	2.9
1.8						34.6	31.1	21.7	13.3	3.1

Table 5.2: Propulsion data.

M	0	0.4	0.8	0.9	1.0	1.2	1.4	1.6	1.8
$c_{L\alpha}$	3.44	3.44	3.44	3.58	4.44	3.44	3.01	2.86	2.44
c_{D0}	0.013	0.013	0.013	0.014	0.031	0.041	0.039	0.036	0.035
η	0.54	0.54	0.54	0.75	0.79	0.78	0.89	0.93	0.93

Table 5.3: Aerodynamic data.

5.2.4 Numerical Solution

Using the minimum curvature approximations for the tabular data, the minimum time to climb problem can be solved using the direct transcription algorithm in SOCS. Table 5.4 summarizes the progress of the algorithm for this application using a linear initial guess for the dynamic variables. The first grid used a trapezoidal discretization (TR) with 10 grid points. The NLP problem was solved using 25 gradient evaluations (GE), 16 Hessian evaluations (HE), and a total of 523 function evaluations (FE) including the finite difference perturbations. The right-hand sides of the ODEs were evaluated 5230 times (NRHS) leading to a solution with a discretization error of $\epsilon_{\max} = 0.35$. Because the error was not sufficiently equidistributed, a second iteration using the trapezoidal discretization was performed. The HSS discretization (HS) was used for the third, fourth, and fifth refinement iterations, after which the HSC method (HC) was used for the remaining refinements. Figure 5.11 illustrates the progress of the mesh-refinement algorithm with the first refinement iteration shaded darkest and the last refinement shaded lightest. For this case, 9 mesh-refinement iterations were required.

Iter.	Disc.	M	GE	HE	FE	NRHS	$\epsilon_{\max}$	CPU (sec)
1	TR	10	25	16	523	5230	0.35×10^{0}	2.8
2	TR	19	8	4	159	3021	0.68×10^{-1}	1.7
3	HS	19	8	5	174	6438	0.87×10^{-2}	3.4
4	HS	37	5	1	74	5402	0.51×10^{-3}	3.7
5	HS	59	4	1	61	7137	0.68×10^{-4}	7.5
6	HC	117	4	1	154	35882	0.12×10^{-4}	11.
7	HC	179	4	1	154	54978	0.13×10^{-5}	16.
8	HC	275	4	1	154	84546	0.14×10^{-6}	27.
9	HC	285	3	1	129	73401	0.97×10^{-7}	22.
Total	-	-	65	31	1582	276035	-	94.88

Table 5.4: Minimum time to climb example.

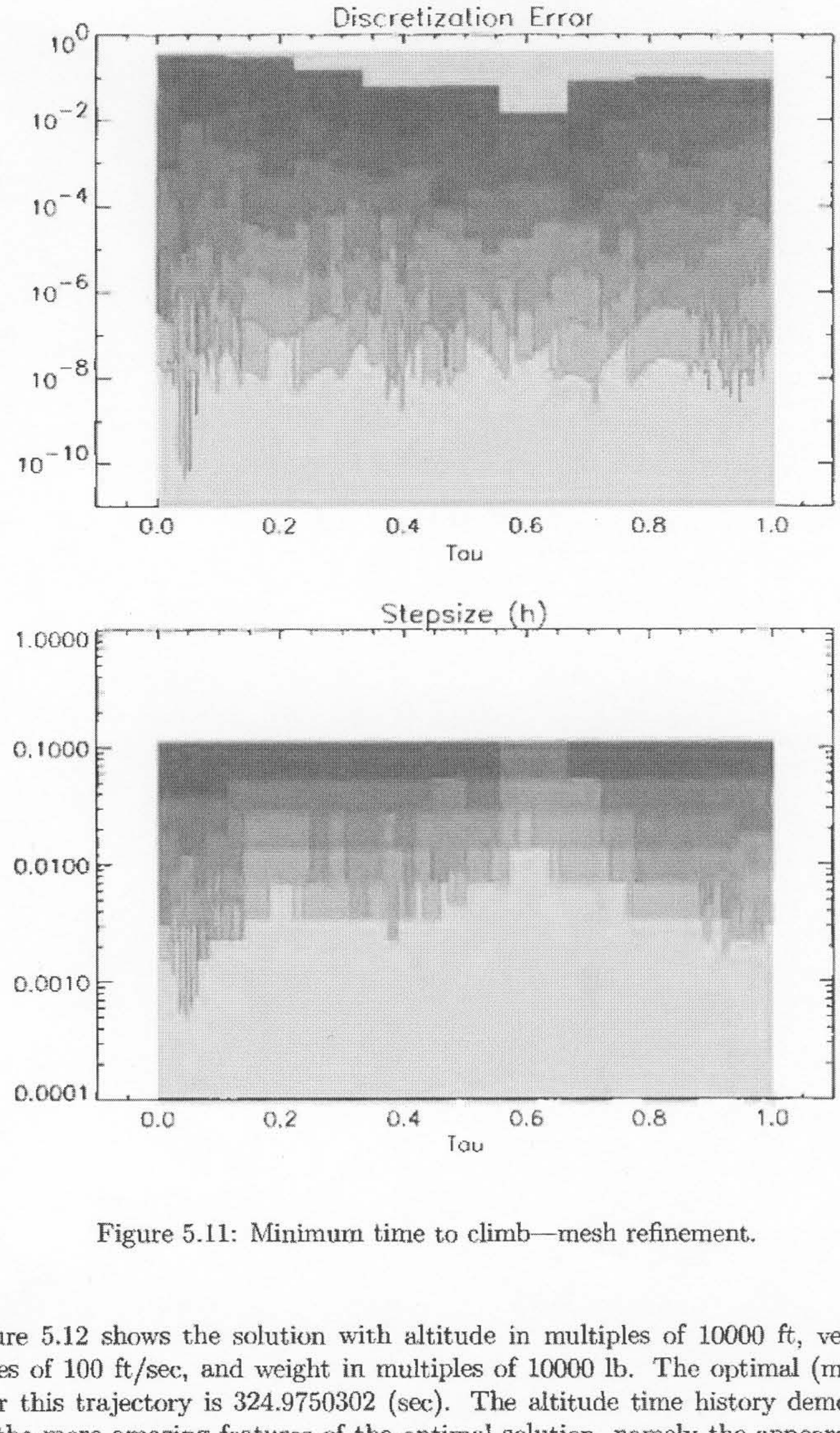

Figure 5.11: Minimum time to climb—mesh refinement.

Figure 5.12 shows the solution with altitude in multiples of 10000 ft, velocity in multiples of 100 ft/sec, and weight in multiples of 10000 lb. The optimal (minimum) time for this trajectory is 324.9750302 (sec). The altitude time history demonstrates one of the more amazing features of the optimal solution, namely the appearance of a *dive* midway through the minimum time to climb trajectory. When first presented in 1969, this unexpected behavior sparked considerable interest and led to the so-called energy-state approach to trajectory analysis. In particular, along the final portion of the trajectory, the energy is nearly constant, as illustrated in the plot of altitude versus velocity.

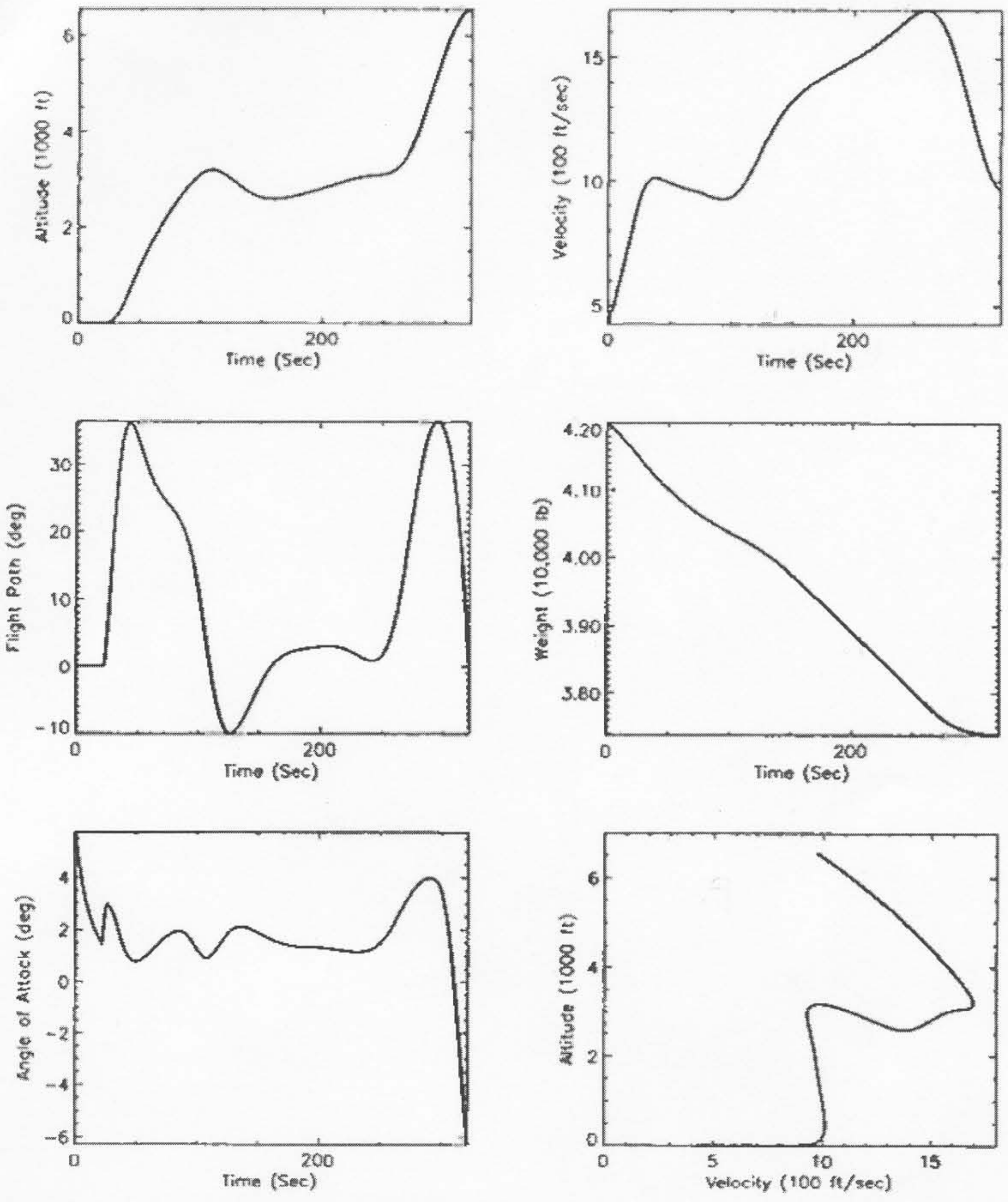

Figure 5.12: Minimum time to climb solution.

[34] A. E. BRYSON, JR., M. N. DESAI, AND W. C. HOFFMAN, *Energy-State Approximation in Performance Optimization of Supersonic Aircraft*, Journal of Aircraft, 6 (1969), pp. 481–488.